A comparative analysis of Painlevé, Lax Pair, and Similarity Transformation methods in obtaining the integrability conditions of nonlinear Schrödinger equations

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We derive the integrability conditions of nonautonomous nonlinear Schrödinger equations using the Lax Pair and Similarity Transformation methods. We present a comparative analysis of these integrability conditions with those of the Painlevé method. We show that while the Painlevé integrability conditions restrict the dispersion, nonlinearity, and dissipation/gain coefficients to be space-independent and the external potential to be only a quadratic function of position, the Lax Pair and the Similarity Transformation methods allow for space-dependent coefficients and an external potential that is not restricted to the quadratic form. The integrability conditions of the Painlevé method are retrieved as a special case of our general integrability conditions. We also derive the integrability conditions of nonautonomous nonlinear Schrödinger equations for two- and three-spacial dimensions.

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I. INTRODUCTION

The question of integrability of nonautonomous nonlinear Schrödinger (NLS) equations has recently been extensively addressed due to their applications in the fields of trapped Bose-Einstein condensates and optical solitons in fibers [1]. Among many works that investigate the integrability of the celebrated NLS equation [1–8], He *et al.* [9] address the most general form of this equation, namely

$$f(x,t)\,\Psi_{xx}(x,t)+g(x,t)\,|\Psi(x,t)|^2\Psi(x,t)+v(x,t)\,\Psi(x,t)+i\,\gamma(x,t)\,\Psi(x,t)+i\,\Psi_t(x,t)=0, \tag{1}$$

where the dispersion coefficient $f(x,t)$, nonlinearity coefficient $g(x,t)$, gain/loss coefficient $\gamma(x,t)$, and external potential $v(x,t)$ are real functions. Applying the Painlevé test, the authors were able to derive the following integrability conditions

$$f(x,t)=f(t), \qquad g(x,t)=g(t), \qquad \gamma(x,t)=\gamma(t), \tag{2}$$

$$v(x,t)=v_0(t)+v_1(t)\,x+v_2(t)\,x^2, \tag{3}$$

where $v_0(t)$ and $v_1(t)$ are arbitrary and $v_2(t)$ is given by

$$4f^3g^2\,v_2+fg(\dot f\dot g+f\ddot g)+g^2(\dot f^2-f\ddot f)-2f^2\dot g^2=0. \tag{4}$$

Here and throughout, the dot represents time derivative and the subscripts represent partial derivatives. Clearly, this shows that, according to the Painlevé test, Eq. (1) is integrable only for time-dependent dispersion, nonlinearity, gain/loss coefficients and for quadratic external potential.

It is well-known, however, that an equation can be integrable in one sense and nonintegrable in another sense. Therefore, we use in this paper two different methods to investigate the integrability of Eq. (1). We use first the Lax Pair method and then the Similarity Transformation method to show that Eq. (1) is indeed integrable for more general time- and space-dependent coefficients. We derive an integrability condition that reproduces the conditions (2)-(4) as a special case. It is found that, while the three methods generate different integrability conditions, when the coefficients of Eq. (1) are space- and time-dependent, the three methods generate the same integrability condition when the coefficients are only time-dependent.

While it is not the aim of this paper to discuss and derive exact solutions of specific examples of integrable NLS equations, we present in section II some examples corresponding to known integrable cases. In addition, we present in sections II and III, as a counter example to the Painlevé result, two cases of NLS equations that are indeed integrable with space- and time-dependent coefficients.

Finally, we use the Lax Pair method, in section IV, to generalize the integrability conditions for NLS equations of higher spacial dimensions. To the best of our knowledge, the integrability conditions derived here (Eqs. (7) and (23)) are presented in the literature for the first time.

II. LAX PAIR METHOD

In the Lax Pair method for solving nonlinear partial differential equations, a pair of 2×2 matrices, $\mathbf{U}$ and $\mathbf{V}$, is first constructed (or derived). The Lax Pair is used to define a linear system of equations for an auxiliary field $\mathbf{\Phi}$, namely $\mathbf{\Phi}_x = \mathbf{U} \cdot \mathbf{\Phi}$ and $\mathbf{\Phi}_t = \mathbf{V} \cdot \mathbf{\Phi}$. The compatibility condition of this system, $\mathbf{U}_t - \mathbf{V}_x + [\mathbf{U}, \mathbf{V}] = 0$, is required to be identical to the nonlinear differential equation. The Darboux transformation [10] is then applied on the linear system to obtain new solutions from known solutions.

We consider the NLS equation to be nonintegrable if the Lax Pair does not exist. Our method of Lax Pair search [11] provides an answer to the question of existence of Lax Pair. In this method, we expand the Lax Pair $\mathbf{U}$ and $\mathbf{V}$ in powers of Ψ and its derivatives with unknown function coefficients and require the compatibility condition to be equivalent to the nonlinear differential equation. This results in a set of coupled differential equations for the unknown coefficients. Solving this system of equations determines the Lax Pair and results in an integrability condition on the functions $f(x,t)$, $g(x,t)$, $\gamma(x,t)$, and $v(x,t)$. The details of this calculation are relegated to Appendix A. The integrability conditions of Eq. (1) turn out to be

$$f(x,t) = \frac{c_1(t)}{g(x,t)^2}, \tag{5}$$

$$\gamma(x,t) = \frac{g_t(x,t)}{g(x,t)} - \frac{1}{2}\frac{\dot{c}_2(t)}{c_2(t)}, \tag{6}$$

and

$$\begin{aligned} & f g^3 \left(f_t \left(g_t - 2g\gamma\right) - f_{tt} g\right) + f_t^2 g^4 + 2 f^3 g^3 \left(g v_{xx} - g_x v_x\right) \\ + \; & f^2 g^2 \left(g \left(4 g_t \gamma + g_{tt}\right) - 2 g_t^2 - 2 g^2 \left(\gamma_t + 2\gamma^2\right)\right) \\ + \; & f^4 \left(36 g_x^4 - 48 g g_{xx} g_x^2 + 10 g^2 g_{xxx} g_x + g^2 \left(6 g_{xx}^2 - g g^{(4)}\right)\right) = 0. \end{aligned} \tag{7}$$

This condition shows that $f(x,t)$, $\gamma(x,t)$ and $v(x,t)$ are determined by $g(x,t)$ and the arbitrary functions $c_1(t)$ and $c_2(t)$. To get specific forms of integrable NLS equations, one starts with a certain form of $g(x,t)$, $c_1(t)$, and $c_2(t)$ from which the functions $f(x,t)$, $\gamma(x,t)$, and $v(x,t)$ will be determined according to Eqs. (5-7). With such a combination of coefficients, Eq. (1) will be integrable. Equations (5-7) represent a main result of the present paper. All previously-known special cases can be derived from these equations. We list here some of these special cases.

Special case I: Constant and linear external potential
With the choices: $g(x,t) = 1$ and $c_1(t) = c_2(t) = 1$, Eq. (1) takes the form

$$\Psi_{xx} + |\Psi|^2\Psi + \Big(c_3(t) + c_4(t)\, x\Big)\, \Psi + i\, \Psi_t = 0, \tag{8}$$

where $c_3(t)$ and $c_4(t)$ are arbitrary real functions arising from integrating Eq. (7) with respect to x. Clearly, with $c_3(t) = c_4(t) = 0$, the well-known homogeneous Gross-Pitaevskii equation (GPE) is obtained. It is also established that Eq. (8) is integrable and exact solutions have been derived [12].

Special case II: Harmonic potential and gain/damping term
For $g(x,t) = 1$, $c_1(t) = 1$, and $c_2(t) = e^{\alpha\, t}$, where α is a real constant, Eq. (1) takes the form

$$\Psi_{xx} + |\Psi|^2\Psi + \left(\frac{\alpha^2}{4}\, x^2 - \frac{\alpha}{2}\, i + c_3(t) + c_4(t)\, x\right)\, \Psi + i\, \Psi_t = 0. \tag{9}$$

With $c_3(t) = c_4(t) = 0$ this will be the typical GPE with an expulsive harmonic potential and damping.

Special case III: Harmonic potential and time-dependent nonlinearity
For $g(x,t) = e^{\alpha\, t}$, $c_1(t) = c_2(t) = e^{2\alpha\, t}$, Eq. (1) takes the form

$$\Psi_{xx} + e^{\alpha\, t}\, |\Psi|^2\Psi + \left(\frac{\alpha^2}{4}\, x^2 + c_3(t) + c_4(t)\, x\right)\, \Psi + i\, \Psi_t = 0. \tag{10}$$

With $c_3(t) = c_4(t) = 0$ this will be the GPE with harmonic potential and nonlinearity growing exponentially with time. This equation was shown to be integrable and exact solitonic solution were obtained by Ref. [4].

Special case IV: x^n-dependent coefficients
For $g(x,t)=x^n$, where n is an integer, and $c_1(t)=c_2(t)=1$, Eq. (1) takes the form

$$x^{-2n}\,\Psi_{xx}+x^n\,|\Psi|^2\Psi+\left(-\frac{1}{4}n(n+2)x^{-2(n+1)}+\frac{c_3(t)}{n+1}\,x^{n+1}+c_4(t)\,x\right)\,\Psi+i\,\Psi_t=0. \tag{11}$$

This is our first counter example to the conclusion of the Painlevé test; an integrable NLS equation with space-dependent coefficients.

Special case V: Time-dependent coefficients
For the special case of $g(x,t)=g(t)$, Eqs. (5) and (6) result in $f(x,t)=f(t)$ and $\gamma(x,t)=\gamma(t)$, where $f(t)$ and $\gamma(t)$ are now independent. In this case, Eq. (7) leads to

$$\begin{aligned} v(x,t) &= \frac{-g^2\dot{f}^2+fg\left(g\ddot{f}+\dot{f}\left(2g\gamma-\dot{g}\right)\right)+f^2\left(2\dot{g}^2-g\left(\ddot{g}+4\gamma\dot{g}\right)+2g^2\left(\dot{\gamma}+2\gamma^2\right)\right)}{4f^3g^2}\,x^2\\ &+\ c_3(t)\,x+c_4(t), \end{aligned} \tag{12}$$

where $c_3(t)$ and $c_4(t)$ are arbitrary. Comparing this expression with $v(x,t)=v_2(t)\,x^2+v_1(t)\,x+v_3$, we get

$$\begin{aligned} &-\ fg^2\ddot{f}+fg\dot{f}\dot{g}-2fg^2\gamma\dot{f}+g^2\dot{f}^2+f^2g\ddot{g}+4f^2g\gamma\dot{g}-2f^2\dot{g}^2\\ &+\ 4v_2f^3g^2-2f^2g^2\dot{\gamma}-4f^2g^2\gamma^2=0. \end{aligned} \tag{13}$$

This is the integrability condition (22) obtained by He *et al.* [9]. For the special case of $\gamma(t)=0$, we get

$$4f^3g^2\,v_2+fg(\dot{f}\dot{g}+f\ddot{g})+g^2(\dot{f}^2-f\ddot{f})-2f^2\dot{g}^2=0, \tag{14}$$

which is the integrability condition (26) of He *et al.*

III. SIMILARITY TRANSFORMATION METHOD

This is a method that is used frequently in the literature to generate new solutions of a nonautonomous NLS equation from those of the standard homogeneous NLS equation [5–7, 13]. By transforming the coordinates and the wave function such that the nonautonomous NLS equation is transformed into the standard NLS equation, one can use the same transformation to obtain solutions of the nonautonomous NLS from the solutions of the standard NLS. Here, we exploit this method to investigate the integrability of NLS equations. It turns out that for the transformation to be possible, the coefficients of the nonautonomous NLS must satisfy certain integrability conditions.

The integrability condition (4) can be obtained by requiring the transformation

$$\Psi(x,t)=\exp\Big(\,\beta(x,t)+i\,\theta(x,t)\Big)\,Q(X(x,t)), \tag{15}$$

where $\beta(x,t)$, $\theta(x,t)$, and $X(x,t)$ being real functions, to transform Eq. (1) into the following time-independent homogeneous equation

$$p(x,t)\left(\,\epsilon\,Q_{XX}(X)+\delta\,|Q(X)|^2\,Q(X)\right)=0, \tag{16}$$

where $p(x,t)$ is in general a complex function that will be determined in terms of $\beta(x,t)$, $\theta(x,t)$, and $X(x,t)$, and ϵ and δ are real constants. Substituting this form of $\Psi(x,t)$ in Eq. (1), it takes the form

$$\begin{aligned} &e^{2\beta}\,g\,Q(X)|Q(X)|^2+f\,X_x^2\,Q''(X)+\left[iX_t+f\ \left(2X_x\left(\beta_x+i\theta_x\right)+X_{xx}\right)\right]\,Q'(X)\\ +\ &\left[v+i\,\gamma+i\,\beta_t-\theta_t+f\left(\left(\beta_x+i\theta_x\right)^2+\beta_{xx}+i\theta_{xx}\right)\right]\,Q(X)=0. \end{aligned} \tag{17}$$

Comparing this equation with Eq. (16), we get 8 coupled differential equations for the unknown functions. These equations can be solved resulting in rather lengthy expressions for the functions $f(x,t)$, $\theta(x,t)$, and $v(x,t)$ in terms

of $\beta(x,t)$ and arbitrary functions of t, as presented in detail in Appendix B. We also show in Appendix B that in the special case of $\beta(x,t) = \beta(t)$ and $\gamma(x,t) = 0$, the expression for $v(x,t)$ simplifies to

$$\begin{aligned} v(x,t) &= \dot{c}_5 + \frac{{c_6}^2 \delta^2 f \dot{c}_2^2}{4\epsilon^2 g^2} \\ &- \frac{c_6 \delta \left(f g \ddot{c}_2 + \dot{c}_2 \left(g \dot{f} - 2 f \dot{g} \right) \right)}{2\epsilon f g^2} x \\ &+ \frac{-f g \left(\dot{f} \dot{g} + f \ddot{g} \right) + g^2 \left(f \ddot{f} - \dot{f}^2 \right) + 2 f^2 \dot{g}^2}{4 f^3 g^2} x^2, \end{aligned} \tag{18}$$

where c_6 is an arbitrary real constant, and $c_2(t)$ and $c_5(t)$ are arbitrary real functions. Equating the coefficient of the x^2-term of the previous equation with v_2, we retrieve the integrability condition Eq. (4) of the Painlevé analysis and the Lax Pair method.

As a second counter example on the integrability of Eq. (1) with time- and space-dependent coefficients, we take the special case of $\beta(x,t) = -(1/2) \log x$ and $\gamma(x,t) = 0$. Substituting this in $f(x,t)$, $g(x,t)$, and $v(x,t)$, Eq. (1) takes the form

$$i\,\Psi_t + \Psi_{xx} + t^2\,x^3\,|\Psi|^2\,\Psi + \left(\frac{3x^2}{16t^2} - \frac{3}{4x^2} \right) \Psi = 0, \tag{19}$$

where we have set, for simplicity, $\delta = \epsilon = 1$. An exact solution of the homogeneous NLS, Eq. (16), is $Q(X) = -\mathrm{sn}(X/\sqrt{2}\,|\,-1)$. Applying the Similarity Transformation, we get a solution of Eq. (19), namely $\Psi(x,t) = \exp\left(-i\,x^2/8\,t\right) \mathrm{sn}(t\,x^2/\sqrt{8}\,|\,-1)/\sqrt{x}$, where $\mathrm{sn}(x\,|\,m)$ is the Jacobian elliptic function of modulus m.

IV. INTEGRABILITY CONDITIONS OF TWO- AND THREE-DIMENSIONAL NLS EQUATIONS

For the one-dimensional NLS equation, Eq. (1), to be integrable, the nonlinearity coefficient, $g(x,t)$, and the dispersion coefficient, $f(x,t)$, should be related according to the integrability condition Eq. (5). It is expected that an additional term proportional to $\Psi_x(x,t)$ in the NLS equation would affect this relation between $g(x,t)$ and $f(x,t)$, such that a wider class of integrable NLS would be obtained. In other words, it is hoped that one-dimensional NLS equations which are nonintegrable to become integrable as a result of this addition. In addition, with this term the NLS corresponds to two- and three-dimensional physical systems. Furthermore, such an addition would allow for the consideration of position-dependent effective mass situations.

We choose the Lax Pair method to investigate the integrability of the following general NLS equation

$$f(x,t)\,\Psi_{xx}(x,t) + h(x,t)\,\Psi_x(x,t) + g(x,t)\,|\Psi(x,t)|^2\Psi(x,t) + v(x,t)\,\Psi(x,t) + i\,\gamma(x,t)\,\Psi(x,t) + i\,\Psi_t(x,t) = 0, \tag{20}$$

where $h(x,t)$ is a real function.

Employing the Lax Pair method in a similar manner as explained in section II and Appendix A, we obtain the following integrability conditions

$$f(x,t) = \frac{c_1(t)\,H(x,t)}{g(x,t)^2}, \tag{21}$$

$$\gamma(x,t) = \frac{g_t(x,t)}{g(x,t)} - \frac{1}{2}\frac{\dot{c}_2(t)}{c_2(t)} - \frac{1}{4}\frac{H_t(x,t)}{H(x,t)}, \tag{22}$$

and

$$\begin{aligned} &- 4 f g^3 H^2 \left(f_t \left(2g\gamma - g_t \right) + f_{tt} g \right) + 4 f_t^2 g^4 H^2 + 4 f^3 g^3 H \left(v_x \left(g H_x - 2 g_x H \right) + 2 g H v_{xx} \right) \\ &+ f^4 \Big[- g^2 H_x \left(3 \left(g g_{xx} - 2 g_x^2 \right) H_x + g g_x H_{xx} \right) - 2 g H \left(48 g_x^3 H_x - 10 g g_x^2 H_{xx} + g g_x \left(g H_{xxx} - 42 g_{xx} H_x \right) \right. \\ &+ \left. 2 g^2 \left(2 g_{xx} H_{xx} + 3 g_{xxx} H_x \right) \right) + 4 \left(36 g_x^4 - 48 g g_{xx} g_x^2 + 10 g^2 g_{xxx} g_x + g^2 \left(6 g_{xx}^2 - g g^{(4)} \right) \right) H^2 \Big] \\ &+ 4 f^2 g^2 H^2 \left[g \left(4 g_t \gamma + g_{tt} \right) - 2 g_t^2 - 2 g^2 \left(\gamma_t + 2\gamma^2 \right) \right) \right] = 0, \end{aligned} \tag{23}$$

where

$$H(x,t) = \exp\left(\int \frac{2h(x,t)}{f(x,t)}\, dx\right). \tag{24}$$

As a first check, we confirm that the above integrability conditions indeed reduce to those of the previous sections, Eqs. (5-7) for $H(x,t) = 1$ or equivalently $h(x,t) = 0$. These integrability conditions give all coefficients of the NLS equation in terms of two arbitrary functions $g(x,t)$ and $h(x,t)$. In the following, we present some interesting special cases.

Special case I: Constant effective mass
For $g^2(x,t) = H(x,t) = x^n$, we get the following integrable NLS equation

$$i\,\Psi_t(x,t) + \Psi_{xx}(x,t) + \frac{n}{x}\,\Psi_x(x,t) + x^n\,|\Psi(x,t)|^2\Psi(x,t) + \left(c_3(t) + c_4(t)\,x + \frac{n(n-2)}{4x^2}\right)\Psi(x,t) = 0, \tag{25}$$

where, $c_3(t)$ and $c_4(t)$ are arbitrary real functions. This equation corresponds to a NLS in $n+1$ spacial dimensions with constant effective mass. Of particular importance, are the cases of $n = 1$ and $n = 2$, corresponding to two and three dimensions, respectively, with power law for the strength of the interatomic interaction, x^n, and an effective potential that includes the centripetal part with a suitably-defined angular momentum.

Special case II: Power-law effective mass
For $g^2(x,t) = x^p$ and $H(x,t) = x^q$, the integrable NLS takes the form

$$\begin{aligned} &i\,\Psi_t(x,t) + x^{q-2p}\Psi_{xx}(x,t) + \frac{1}{2}\,q\,x^{q-2p-1}\,\Psi_x(x,t) + x^p\,|\Psi(x,t)|^2\Psi(x,t) \\ +\; &\left(c_3(t) + \frac{2c_4(t)}{2+2p-q}\,x^{1+p-q/2} - \frac{1}{4}\,p(2+p-q)\,x^{q-2p-2}\right)\Psi(x,t) = 0. \end{aligned} \tag{26}$$

The position-dependent effective mass appears in the NLS equation through the second derivative operator, which in this case will be given explicitly by

$$-\frac{d}{dx}\left(\frac{1}{2m(x)}\frac{d\Psi(x,t)}{dx}\right) = x^{q-2p}\Psi_{xx}(x,t) + \frac{1}{2}\,q\,x^{q-2p-1}\,\Psi_x(x,t). \tag{27}$$

This can be satisfied for $q = 4p$, which corresponds to an effective mass $m(x) = -x^{-2p}/2$, and Eq. (28) simplifies to

$$\begin{aligned} &i\,\Psi_t(x,t) + x^{2p}\Psi_{xx}(x,t) + 2p\,x^{2p-1}\,\Psi_x(x,t) + x^p\,|\Psi(x,t)|^2\Psi(x,t) \\ +\; &\left(c_3(t) + \frac{c_4(t)}{1-2p}\,x^{1-p} - \frac{1}{4}\,p\,(2-3p)\,x^{2(p-1)}\right)\Psi(x,t) = 0. \end{aligned} \tag{28}$$

V. CONCLUSIONS AND OUTLOOK

Using the Lax Pair and Similarity Transformation methods, we have shown that, in contrast to the results of the Painlevé analysis, Eq. (1) is integrable for time- and space-dependent coefficients. This is of course not surprising since it is well-known that an equation can be integrable in one sense and nonintegrable in another sense [14]. For the special case when the coefficients of Eq. (1) are only time-dependent, the integrabilty conditions of the three methods become identical.

The general integrability conditions found here, Eqs. (7) and (23), would be useful for cases of space-dependent dispersion such as optical solitons propagating in a medium of space-dependent refractive index.

It is in the nature of the Lax Pair method that new solutions are obtained using old (seed) solutions. Therefore, the integrability conditions imposed by the Lax Pair method are conditions on the possibility of mapping solutions of the nonautonomous NLS equation into other solutions of the same NLS. On the other hand, the integrability conditions imposed by the Similarity Transformation method are conditions on the possibility of mapping solutions of the homogeneous NLS equation into those of the nonautonomous NLS. The integrability conditions of the Painlevé method have even more fundamentally different origin related to the *movable* singularities of the NLS equation. The fact that these three fundamentally different methods generate the same integrability condition for the case of time-dependent coefficients of the NLS equation, suggests two illuminating ideas which require further investigation. First,

the three seemingly fundamentally different methods may be after all not so different such that they can be unified in a single transformation method. Secondly, it seems that - at least for the case of time-dependent coefficients - integrability is an intrinsic property of the NLS equation, which is independent of the method used to reveal it.

Finally, it is worth mentioning that all of the complicated calculations, including the Lax Pair search, simplifying, and solving the compatibility equations, that led to the results of this paper were facilitated using symbolic programming with the computer program Mathematica 7 [15].

Appendix A: Integrability conditions of the Lax Pair method

The Lax Pair $\mathbf{U}$ and $\mathbf{V}$ are expanded in powers of $\Psi(x,t)$ and its derivatives, as follows

$$\mathbf{U} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix},$$

where
$u_{11} = f_1(x,t) + f_2(x,t)\,\Psi(x,t)$,
$u_{12} = f_3(x,t) + f_4(x,t)\,\Psi(x,t)$,
$u_{21} = f_5(x,t) + f_6(x,t)\Psi^*(x,t)$,
$u_{22} = f_7(x,t) + f_8(x,t)\Psi^*(x,t)$,
$v_{11} = g_1(x,t) + g_2(x,t)\,\Psi(x,t) + g_3(x,t)\,\Psi_x(x,t) + g_4(x,t)\Psi(x,t)\,\Psi^*(x,t)$,
$v_{12} = g_5(x,t) + g_6(x,t)\,\Psi(x,t) + g_7(x,t)\,\Psi_x(x,t) + g_8(x,t)\Psi(x,t)\,\Psi^*(x,t)$,
$v_{21} = g_9(x,t) + g_{10}(x,t)\Psi^*(x,t) + g_{11}(x,t)\,\Psi^*_x(x,t) + g_{12}(x,t)\,\Psi(x,t)\Psi^*(x,t)$,
$v_{22} = g_{13}(x,t) + g_{14}(x,t)\Psi^*(x,t) + g_{15}(x,t)\,\Psi^*_x(x,t) + g_{16}(x,t)\Psi(x,t)\,\Psi^*(x,t)$,
and $f_{1-8}(x,t)$ and $g_{1-16}(x,t)$ are unknown functions.

The compatibility condition

$$\mathbf{U}_t - \mathbf{V}_x + [\mathbf{U},\mathbf{V}] = \mathbf{0} = \begin{pmatrix} 0 & p_1(x,t)\,F[\Psi] \\ p_2(x,t)\,F^*[\Psi^*] & 0 \end{pmatrix}, \tag{A1}$$

with $p_1(x,t)$ and $p_2(x,t)$ being unknown functions, requires

$f_2 = g_3 = f_8 = g_{15} = g_8 = g_{12} = g_{14} = g_2 = f_3 = f_5 = g_9 = g_5 = 0, f_4 = i\,p_1, f_6 = -i\,p_2, g_7 = -f\,p_1, g_{11} = -f\,p_2, g_4 = -g_{16} = -if\,p_1\,p_2$,

and

$$f_{1t} - g_{1x} = 0, \tag{A2}$$

$$f_{7t} - g_{13x} = 0, \tag{A3}$$

$$2f\,p_1\,p_2 + g = 0, \tag{A4}$$

$$f_x\,p_1 - f\,p_1(f_1 - f_7) + f\,p_{1x} - g_6 = 0, \tag{A5}$$

$$f_x p_2 + f\,p_2(f_1 - f_7) + f\,p_{2x} - g_{10} = 0, \tag{A6}$$

$$g_6(f_1 - f_7) - ip_1(g_1 - g_{13} - iv + \gamma) - g_{6x} + ip_{1t} = 0, \tag{A7}$$

$$g_{10}(f_1 - f_7) + ip_2(g_1 - g_{13} - iv - \gamma) + g_{10\,x} + ip_{2\,t} = 0, \tag{A8}$$

$$(f\, p_1\, p_2)_x + g_{10}\, p_1 + g_6\, p_2 = 0. \tag{A9}$$

1. Deriving a relation between $f(x,t)$ and $g(x,t)$

Substituting for $f\, p_1\, p_2$ from Eq. (A4) in Eq. (A9), the latter simplifies to

$$-\frac{1}{2} g_x + g_{10}\, p_1 + g_6\, p_2 = 0. \tag{A10}$$

Multiplying Eq. (A5) by p_2 and Eq. (A6) by p_1, and then adding, we get

$$\begin{aligned} & p_2\,(f\, p_1)_x + p_1 (f\, p_2)_x - p_2\, g_6 - p_1\, g_{10} \\ = \; & (p_1\, p_2\, f)_x + p_1\, p_2\, f_x - p_2\, g_6 - p_1\, g_{10} = 0. \end{aligned} \tag{A11}$$

Substituting for $p_1\, p_2$, $(f\, p_1\, p_2)_x$, and $p_2\, g_6 + p_1\, g_{10}$ from Eqs. (A4), (A9) and (A10), respectively, the last equation takes the form

$$\frac{f_x}{f} = -2\,\frac{g_x}{g}, \tag{A12}$$

which results in

$$f(x,t) = \frac{c(t)}{g(x,t)^2}, \tag{A13}$$

where $c(t)$ is arbitrary.

2. Deriving a relation between $v(x,t)$ and $g(x,t)$

Multiplying Eq. (A5) by p_2 and Eq. (A6) by p_1, and then subtracting, we get

$$f_1 - f_7 = -\frac{g_6}{f\, p_1} + \frac{1}{2}\frac{\partial}{\partial x} \log \frac{p_1}{g} + \frac{f\, p_1\, p_{2x}}{g}. \tag{A14}$$

Multiplying Eq. (A7) by p_2 and Eq. (A8) by p_1, and then subtracting, we get

$$g_6 = \frac{p_1}{g}\left[c_1 + i\, k_{1i} - \frac{c\, g_x}{2\, g^2} + \frac{i}{2}\int \left(3\, g_t - \frac{(2c\,\gamma + \dot{c})\, g}{c}\right) dx\right]. \tag{A15}$$

To obtain the last equation in this form, we have substituted for f_1, g_{10}, p_2, and f from Eqs. (A14), (A10), (A4), and (A13), respectively. Here $c_1(t)$, $k_{1r}(t)$, and $k_{1i}(t)$ are arbitrary real functions that arise from two integrations over x. The last two being the real and imaginary parts of a complex function. The function $c_1(t) + k_{1r}(t)$ appearing in the last equation represents another arbitrary constant. Therefore, we can set, without loss of generality, $k_{1r}(t) = 0$.

Multiplying Eq. (A7) by p_2 and Eq. (A8) by p_1, and then adding, we get

$$g_1 - g_{13} = i\, v + \frac{i\, g^2\, g_6^2}{c\, p_1^2} + \frac{\dot{c}}{2\, c} + \frac{2\, g\, p_{1\,t} + 2\, i\, g_6\, g_x - 3\, p_1\, g_t}{2\, g\, p_1} - \frac{i\, c\, (g\, g_{xx} - 2\, g_x^2)}{2\, g^4}. \tag{A16}$$

To obtain the last equation in this form, we have substituted for f_1, g_{10}, p_2, and f from Eqs. (A14), (A10), (A4), and (A13), respectively. Subtracting Eq. (A3) from Eq. (A2), and substituting for $f_1 - f_7$ and $g_1 - g_{13}$ from Eqs. (A14)

and (A16), we get

$$\begin{aligned}
v_x(x,t) &= -\frac{c}{2g}\frac{\partial^3}{\partial x^3}\frac{1}{g} + \frac{2k_{1i}g_t}{c} - \frac{g}{c}(2k_{1i}\gamma + \dot{k}_{1i}) \\
&+ \frac{g_t - g\gamma}{c}\int (3g_t - \frac{g}{c}(2c\gamma + \dot{c}))\,dx \\
&+ \frac{g}{2c^3}\int \left(c\left(g_t\left(2c\gamma + \dot{c}\right) - 3cg_{tt}\right) + g\left(c\left(2c\gamma_t + \ddot{c}\right) - \dot{c}^2\right)\right)\,dx \\
&+ i\left(\frac{g}{c}(2c_1\gamma + \dot{c}_1) - \frac{2}{c}c_1 g_t\right).
\end{aligned} \tag{A17}$$

Since v is assumed to be real, the imaginary part of the last equation must vanish. This results in the following condition on γ

$$(2c_1\gamma + \dot{c}_1)\; g - 2c_1 g_t = 0, \tag{A18}$$

which gives

$$\gamma(x,t) = \frac{g_t(x,t)}{g(x,t)} - \frac{1}{2}\frac{\dot{c}_1(t)}{c_1(t)}. \tag{A19}$$

Substituting this expression for $\gamma(x,t)$ in Eq. (A17), results in following integrability condition for $v_x(x,t)$

$$\begin{aligned}
v_x(x,t) &= \frac{g}{2c^3{c_1}^2}\left[c^2 c_1\dot{c}_1\int\left(g(\frac{\dot{c}_1}{c_1} - \frac{\dot{c}}{c}) + g_t\right)\,dx\right. \\
&+ \left.\int\left({c_1}^2(c(\dot{c}g_t - cg_{tt}) + g(c\ddot{c} - \dot{c}^2)) + c^2\dot{c}_1^2 g - c^2 c_1(\ddot{c}_1 g + \dot{c}_1 g_t)\right)\,dx\right] \\
&- \frac{c}{2\,g}\frac{\partial^3}{\partial x^3}\frac{1}{g} - \frac{g(c_1\dot{k}_{1i} - k_{1i}\dot{c}_1)}{c\,c_1}.
\end{aligned} \tag{A20}$$

This condition gives $v_x(x,t)$ in terms of $c(t)$, $c_1(t)$ and $g(x,t)$. Using Eqs. (A13) and (A19) to substitute for $c(t)$ and $c_1(t)$, the last equation gives $v_x(x,t)$ in terms of $f(x,t)$, $\gamma(x,t)$, $g(x,t)$ and the arbitrary function $k_{1i}(t)$

$$\begin{aligned}
v_x(x,t) &= \frac{1}{2f^3g^5}\int g^4\left(f\left(g_t\left(f_t + 2f\gamma\right) - fg_{tt}\right) + g\left(f\left(f_{tt} + 2f\gamma_t\right) - f_t^2\right)\right)\,dx \\
&+ 2f^2g^3\left(g_t - g\gamma\right)\int(g(-\frac{f_t}{f} - 2\gamma) + g_t)\,dx \\
&- \frac{1}{2}fg\frac{\partial^3}{\partial x^3}\frac{1}{g} - \frac{2g(2k_{1i}\gamma + \dot{k}_{1i}) - 4k_{1i}g_t}{2fg^2}.
\end{aligned} \tag{A21}$$

Combining the two integrals in Eq. (A20), differentiating with respect to x, and substituting for c and c_1 in terms of f, g and γ, the last condition takes the form

$$\begin{aligned}
& fg^3\left(f_t\left(g_t - 2g\gamma\right) - f_{tt}g\right) + f_t^2 g^4 + 2f^3g^3\left(gv_{xx} - g_x v_x\right) \\
+ \;& f^2g^2\left(g\left(4g_t\gamma + g_{tt}\right) - 2g_t^2 - 2g^2\left(\gamma_t + 2\gamma^2\right)\right) \\
+ \;& f^4\left(36g_x^4 - 48gg_{xx}g_x^2 + 10g^2g_{xxx}g_x + g^2\left(6g_{xx}^2 - gg^{(4)}\right)\right) = 0,
\end{aligned} \tag{A22}$$

which is identical to Eq. (7).

To summarize: We have derived three integrability conditions:

1. Condition (A13) gives $f(x,t)$ in terms of $g(x,t)$ and an arbitrary function $c(t)$.

2. Condition (A19) gives $\gamma(x,t)$ in terms of $g(x,t)$ and an arbitrary function $c_1(t)$.

3. Condition (A22) gives $v_x(x,t)$ in terms of $g(x,t)$, $f(x,t)$ and $\gamma(x,t)$.
The three arbitrary functions $c(t)$, $c_1(t)$, and $k_{1i}(t)$ are independent from each other.

Appendix B: Integrability conditions of the Similarity Transformation method

Comparing Eq. (16) with Eq. (17) we get

$$e^{2\beta}\, g = \delta\, p, \tag{B1}$$

$$f\, X_x^2 = \epsilon\, p, \tag{B2}$$

$$X_t + 2f\, X_x \theta_x = 0, \tag{B3}$$

$$2X_x\, \beta_x + X_{xx} = 0, \tag{B4}$$

$$\beta_t + \gamma + f\,(2\beta_x\, \theta_x + \theta_{xx}) = 0, \tag{B5}$$

$$v - \theta_t + f\left(\beta_x^2 - \theta_x^2 + \beta_{xx}\right) = 0. \tag{B6}$$

Solving Eq. (B4) for X, we find

$$X(x,t) = c_1(t) + c_2(t) \int e^{-2\beta(x,t)}\, dx. \tag{B7}$$

Eliminating $p(x,t)$ from Eqs. (B1) and (B2) and substituting for $X(x,t)$ from the previous equation, we find a compatability condition that combines $f(x,t)$ and $g(x,t)$, namely

$$g(x,t) = \frac{\delta}{\epsilon}\, c_2(t)^2\, e^{-6\beta(x,t)}\, f(x,t). \tag{B8}$$

Solving Eq. (B3) for $\theta(x,t)$, we get

$$\theta(x,t) = -\int \frac{e^{2\beta}\left(\int\, e^{-2\beta}\left(\dot{c}_2 - 2\, c_2\beta_t\right)\, dx + \dot{c}_1\right)}{2\, c_2 f}\, dx + c_\theta(t), \tag{B9}$$

where $c_\theta(t)$ is arbitrary. Solving Eq. (B5) for $f(x,t)$, we get

$$f(x,t) = c_f(t) \exp\left(\int \frac{e^{-2\beta}\left(4e^{2\beta}\beta_x\left(\int e^{-2\beta}\left(\dot{c}_2 - 2c_2\beta_t\right)\, dx + \dot{c}_1\right) - 4c_2\beta_t + \dot{c}_2 - 2\, c_2\, \gamma\right)}{\int e^{-2\beta}\left(\dot{c}_2 - 2c_2\beta_t\right)\, dx + \dot{c}_1}\, dx\right), \tag{B10}$$

where $c_f(t)$ is arbitrary.

1. Special case I

To obtain the special case of time-dependent coefficients of the NLS equation, namely $f(x,t) = f(t)$, $g(x,t) = g(t)$, we assume $\beta(x,t) = \beta(t)$ and $\gamma(x,t) = \gamma(t)$. With these assumptions, Eq. (B10) takes the form

$$f(x,t) = \left((2c_2\, \dot{b} - \dot{c}_2)\, x - e^{2b}\dot{c}_1\right)^{\frac{2c_2(2\dot{b}+\gamma)-\dot{c}_2}{2c_2\dot{b}-\dot{c}_2}}. \tag{B11}$$

The function $f(x,t)$ becomes x-independent either by setting the exponent in the previous equation to zero or by setting the coefficient of x to zero. We notice that the coefficient of x is identical with the denominator of the exponent. Therefore, we take the first choice. Setting the exponent to zero and solving for c_2, we get

$$c_2(t) = c_8\, e^{4\, b + 2\int \gamma\, dt}. \tag{B12}$$

Substituting this expression for $c_2(t)$ in Eqs. (B10) and (B9), we get $f(x,t) = c_f(t) \equiv f(t)$ and

$$\theta(x,t) = c_\theta - \frac{x\left(x\dot{b} + \dot{c}_1\, e^{-2(b+\int \gamma\, dt)}/c_8\right) + x^2\gamma}{2f}. \tag{B13}$$

Finally, the function $v(x,t)$ can be obtained from Eq. (B6)

$$\begin{aligned} v(x,t) &= \frac{e^{-2(b+\int\gamma\,dt)}\left(\dot{c}_1\left(4f\left(\dot{b}+\gamma\right)+\dot{f}\right)-f\ddot{c}_1\right)}{2c_8f^2}x \\ &+ \frac{\dot{f}\left(\dot{b}+\gamma\right)+f\left(-\ddot{b}+4\gamma\dot{b}+2\dot{b}^2-\dot{\gamma}+2\gamma^2\right)}{2f^2}x^2 \\ &+ \frac{\dot{c}_1^2e^{-4(b+\int\gamma\,dt)}}{4{c_8}^2f}+\dot{c}_\theta \end{aligned} \tag{B14}$$

taking the special case $\gamma=0$ and substituting from Eq. (B8) for $b=(1/2)\log\left(\epsilon\,g/c_8^2\,\delta\,f\right)$, the last equation gives

$$\begin{aligned} v(x,t) &= \dot{c}_\theta+\frac{{c_8}^2\delta^2f\dot{c}_1^2}{4\epsilon^2g^2} \\ &- \frac{c_8\delta\left(fg\ddot{c}_1+\dot{c}_1\left(g\dot{f}-2f\dot{g}\right)\right)}{2\epsilon fg^2}x \\ &+ \frac{-fg\left(\dot{f}\dot{g}+f\ddot{g}\right)+g^2\left(f\ddot{f}-\dot{f}^2\right)+2f^2\dot{g}^2}{4f^3g^2}x^2 \end{aligned} \tag{B15}$$

Equating the coefficient of the x^2-term with v_2, we retrieve the integrability condition Eq. (4) of the Painlevé analysis and the Lax Pair method.

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